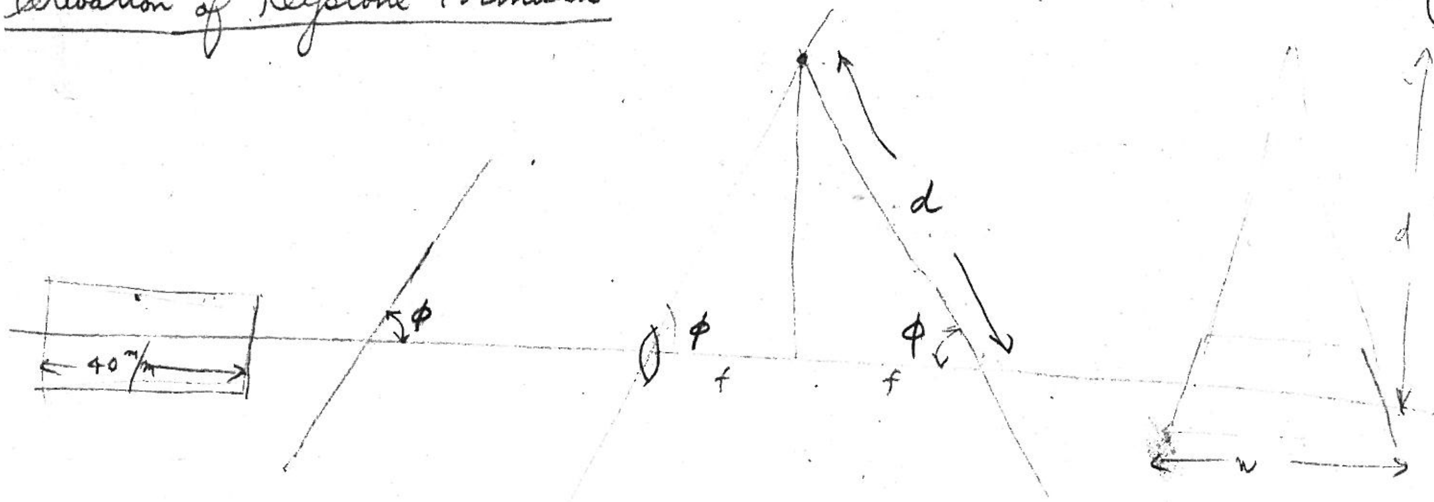
Derivation of Keystone Formula




$$d = f \sec \phi$$

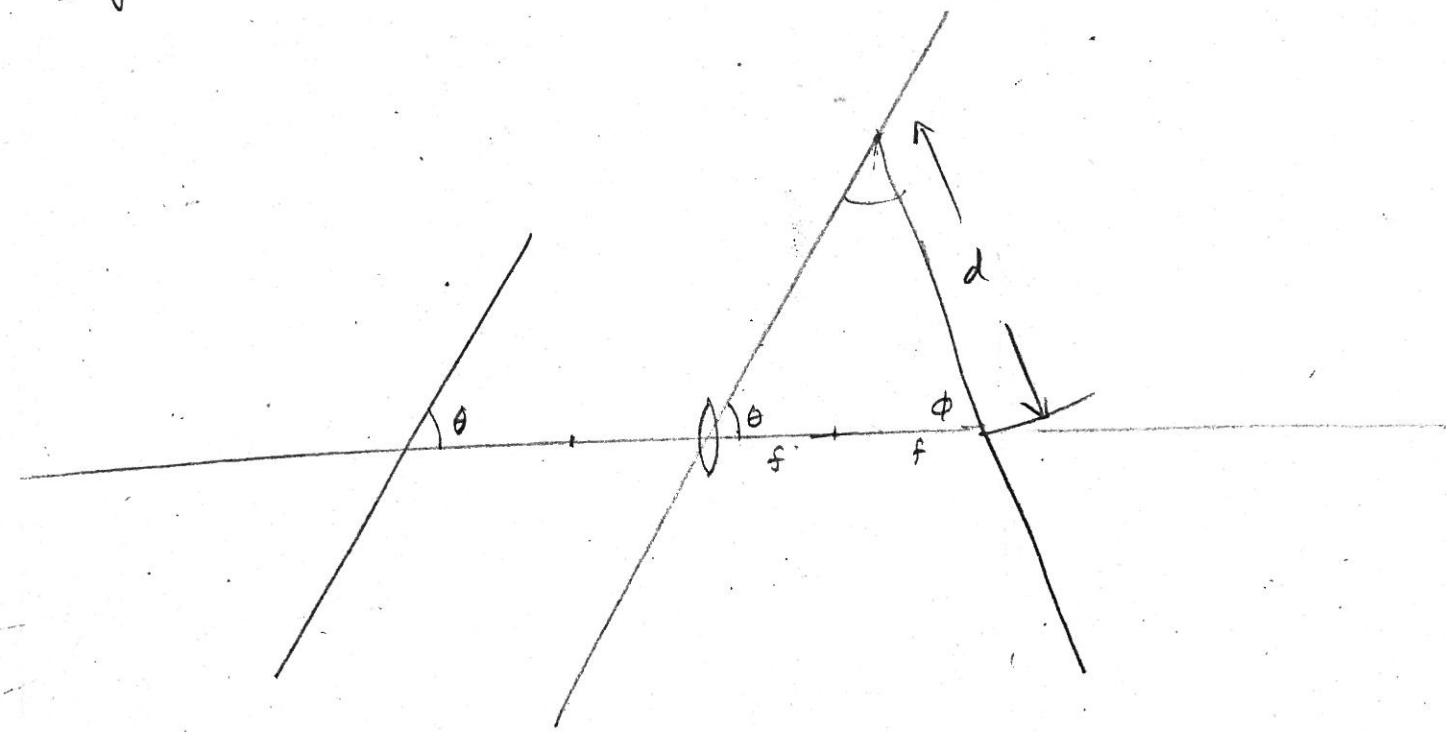
$$w = 40 \text{ mm}$$

For small angles: $\alpha = \frac{w}{d} = \frac{40}{f} \cos \phi$

$= 1.14 \cos \phi$ for 35 m/m lens,
$= 0.80 \cos \phi$ for 50 m/m
$= 0.18 \cos \phi$ for $8\frac{5}{8}$" lens

For large angles: $\tan \frac{\alpha}{2} = \frac{w}{2d} = \frac{w}{2f} \cos \phi$

## Keystone due to unequal tilts




$$\frac{d}{\sin\theta} = \frac{2f}{\sin\left(180^\circ - (\theta+\phi)\right)}$$

$$d = 2f\,\frac{\sin\theta}{\cancel{\sin(\theta+\phi)}}$$

$$\alpha = \frac{W}{d} = \frac{W}{2f}\,\frac{\sin(\theta+\phi)}{\sin\theta}$$

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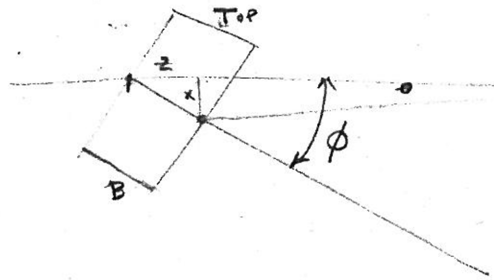
also

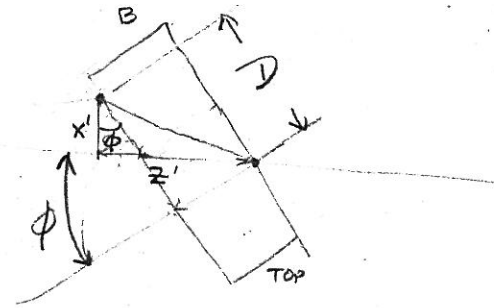
$$\sin\theta+\phi = \sin\theta\cos\phi + \cos\theta\sin\phi$$

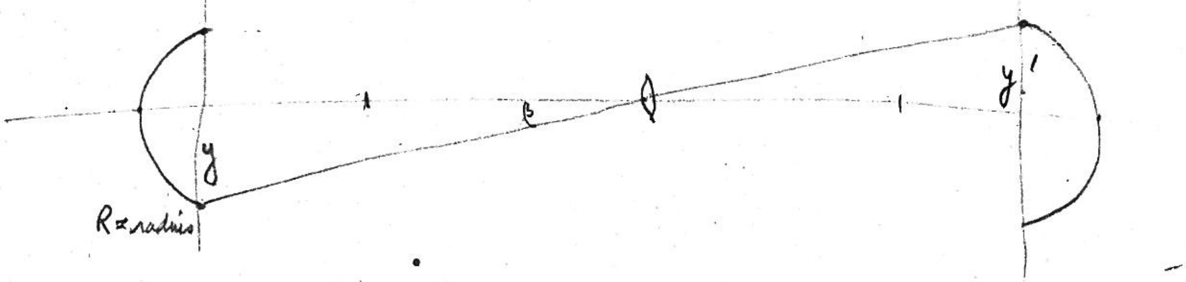
$$\therefore \quad \alpha = \frac{W}{2f}\left[\cos\phi + \cos\theta\,\frac{\sin\phi}{\sin\theta}\right]$$

$$\alpha = \frac{W}{2f}\left[\cos\phi + \frac{s}{s'}\cos\theta\right]$$

Derivation: [Droop for equal tilt and both surfaces concave (toward lens]




$$S = \text{sag} = \frac{y^2}{2R} \qquad z = S \cos\phi$$

$$x = S \sin\phi = \frac{y^2}{2R}\sin\phi$$

$$y = y = R \sin\frac{W}{2R}$$

$$z = S\cos\phi = \frac{y^2}{2R}\cos\phi$$

Independent of f' if width of picture is preserved then

$$y' = y \quad \text{hence } S'(\text{new sag}) = S$$

and $\therefore$ $x' = x$

$$z' = x\tan\phi + S\sec\phi$$

$$D = x\sec\phi + S\tan\phi$$

$$D = \frac{y^2}{2R}\sin\phi\sec\phi + \frac{y^2}{2R}\tan\phi = \frac{2y^2}{2R}\tan\phi = \frac{y^2}{R}\tan$$

Example #1: $R = 50\ mm$

$\frac{W}{2} = 20\ mm$ ($\frac{1}{2}$ width of porch)

$$y = 50 \sin \frac{20}{50} = 50 \sin 23^\circ = 50\,(.392) = 19.6$$

$$D = \frac{y^2}{R} \tan\phi = \frac{(19.6)^2}{50} \tan 20^\circ = 2.8\ mm$$

which is too low should be 3.5 mm

$$\tan\phi = \frac{D}{\left(\frac{y^2}{R}\right)} = \frac{3.5}{7.68} = .456$$

$$\phi = 24.5^\circ$$

Keystone Formula

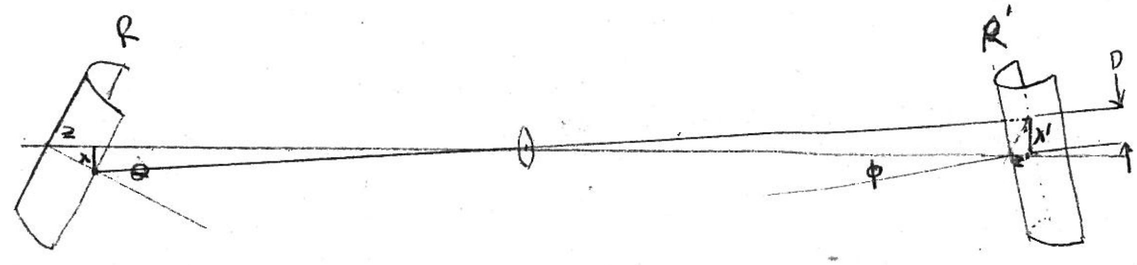
for $8\frac{5}{8}''$ lens: $\alpha = 0.18 \cos\phi$

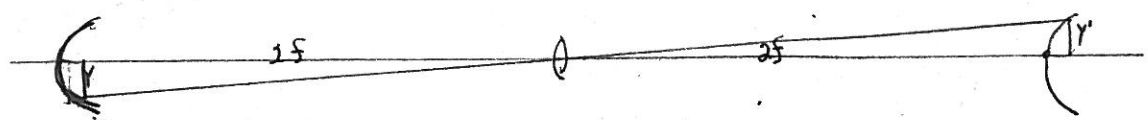
$$\alpha = 0.18\,(.91) = +.9.3^\circ$$ which is nearly correct.

for 35 m/m

$$\alpha = 1.14 \cos\phi$$
$$= 1.14\,(.91) = 59.5^\circ$$

Derivation: [Droop for unequal tilt and object concave and image convex toward lens]



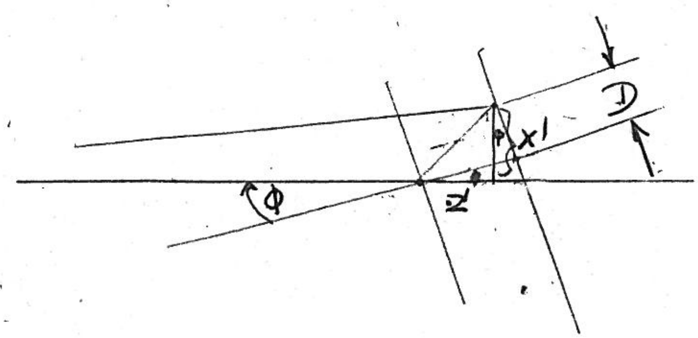




$$S' = S = \frac{y^2}{2R}$$

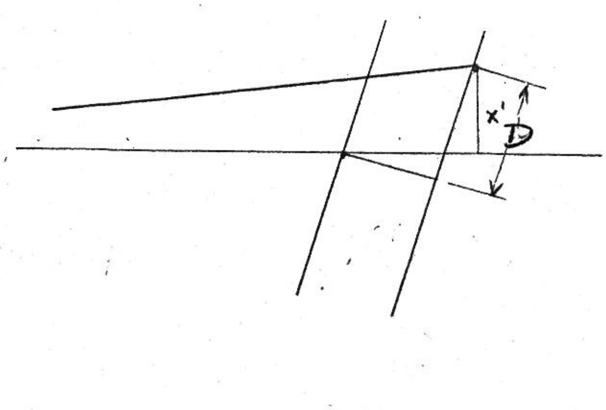
$$x = S \sin\theta = \frac{y^2}{2R}\sin\theta$$

$$y = x \quad = R \sin\frac{W}{2R}$$

$$z = S\cos\theta = \frac{y^2}{2R}\cos\theta$$




let $y' = y$, $x' = x$ so $S' = S$




$$z' = -S'\sec\phi + x'\tan\phi$$

$$D = x'\sec\phi - S'\tan\phi$$

$$= \frac{y^2}{2R_1}\sin\theta\sec\phi - \frac{y^2}{2R'}\tan\phi$$

$$= \frac{y^2}{2R}\left(\sin\theta\sec\phi - \tan\phi\right)$$

<u>Example # 2</u>: $R = 50$ mm $\qquad f' = 50$ m/m

$\frac{W}{2} = 20$ mm

$y = 19.6$ mm

$$D = \frac{y^2}{2R}\left(\sin\theta \sec\phi - \tan\phi\right)$$

assume: $\theta = 20°$

To get $\alpha = 9°$

$$\sin(\theta+\phi) = \frac{2f \cdot \alpha \sin\theta}{W} = \frac{100}{40}\,\frac{9}{57.3}\,(.342)$$

$$\sin(\theta+\phi) = .134 = \sin 7.7°$$

$$\theta+\phi = 7.7°$$

$$\phi = -12.3°$$

$$\therefore D = \frac{(19.6)^2}{100}\left[(.342)(1.02) + .218\right]$$

$.350 + .218$

$.768$

$D = 2.95$ m/m

Example # 3:

$R = 50 \text{ mm}$

$\frac{W}{2} = 20 \text{ mm}$

$\theta = 30^\circ$

$\sin(\theta+\phi) = \left(\frac{100}{40} \ \frac{9}{57.3}\right)(.5)$

$\theta+\phi = .196 = \sin \ 11.3^\circ$

$\phi = 11.3 - 30$

$\phi = -18.7^\circ$

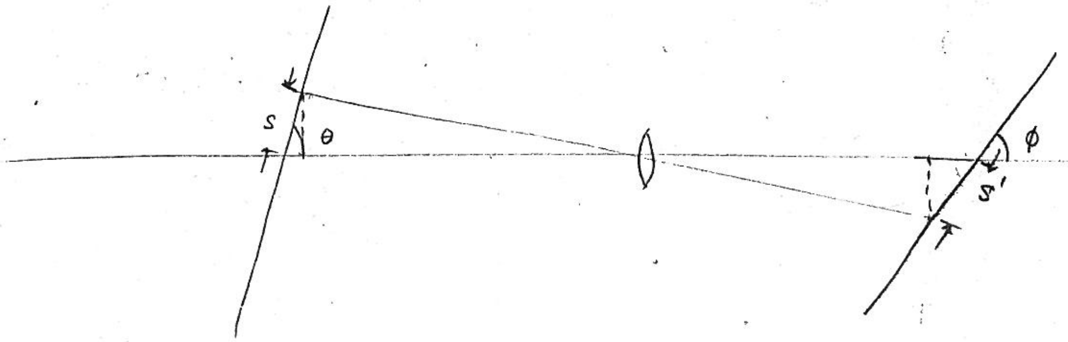
$D = \frac{(19.6)^2}{100}\left[(.5)(1.06) + .338\right]$

$.530 + .338$

$.868$

$D = 3.33 \text{ mm}$

Fore-shortening in image




$$s' = s \frac{\sin\theta}{\sin\phi} \qquad \frac{s'}{s} = \frac{\sin\theta}{\sin\phi}$$

$$s' = s \frac{\cos 90-\theta}{\cos 90-\phi} \qquad \frac{s'}{s} = \frac{\cos 90-\theta}{\cos 90-\phi}$$

$$\frac{s'}{s} = \frac{\cos 30^\circ}{\cos 18.7^\circ} = \frac{.866}{.947} = .915 \rightarrow 8.9\%$$

Ratio $\dfrac{\text{Dist between posts}}{\text{Height of center post}} = \dfrac{20}{10} = 2$

Was originally $= \dfrac{91.5}{53} = 1.73$

$$\frac{2-1.73}{1.73} = 15.6\%$$